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Probabilistic bounds on the length of a longest edge in Delaunay graphs of random points in d-dimensions ☆

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ABSTRACT

Motivated by low energy consumption in geographic routing in wireless networks, there has been recent interest in determining bounds on the length of edges in the Delaunay graph of randomly distributed points. Asymptotic results are known for random networks in planar domains. In this paper, we obtain upper and lower bounds that hold with parametric probability in any dimension, for points distributed uniformly at random in domains with and without boundary. The results obtained are asymptotically tight for all relevant values of such probability and constant number of dimensions, and show that the overhead produced by boundary nodes in the plane holds also for higher dimensions. To our knowledge, this is the first comprehensive study on the lengths of long edges in Delaunay graphs.

1. Introduction

We study the length of a longest Delaunay edge for points randomly distributed in multidimensional Euclidean spaces. In particular, we consider the Delaunay graph for a set of n points distributed uniformly at random in a d-dimensional body of unit volume. It is known that the probability that uniformly distributed random points are not in general position[1] is negligible and therefore it is safe to focus on generic sets of points [3], which we do throughout the paper.

The motivation to study such settings comes from the Random Geometric Graph (RGG) model in which n nodes are distributed uniformly at random in a disk or, more generally, according to some specified density function on d-dimensional Euclidean space [4]. The problem has attracted recent interest because of its applications in energy-efficient geometric routing and flooding in wireless sensor networks (see, e.g., [5–8]).

Related work For n random points uniformly chosen from the unit disk, Kozma, Lotker, Sharir, and Stupp [6] show that the asymptotic length of a longest Delaunay edge depends on the distances of the endpoints from the disk boundary. More

☆ Earlier and partial versions of this work have appeared in [1,2].

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[1] A set of $d+1$ points in d-dimensional Euclidean space is said to be *in general position* if no hyperplane contains all of them. We say that such a set is *generic*, or *degenerate* otherwise.

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Table 1
Summary of results in asymptotic notation for constant d.

	w.p. $\geq 1-\varepsilon$	w.p. $\geq \varepsilon$
Surface of spherical cap whose orthodromic diameter is a Delaunay edge, when points are sampled from the surface of a d-sphere.	$O(\frac{\log(n/\varepsilon)}{n})$	$\Omega(\frac{\log(1/\varepsilon)}{n+\log(1/\varepsilon)})$
Volume of ball cap whose base diameter is a Delaunay edge, when points are sampled from a d-ball.	$O(\frac{\log(n/\varepsilon)}{n})$	$\Omega(\frac{\log(1/\varepsilon)}{n+\log(1/\varepsilon)})$

specifically, let σ be the sum of these two distances; their bounds are $O(\sqrt[3]{(\log n)/n})$ if $\sigma \leq ((\log n)/n)^{2/3}$, $O(\sqrt{(\log n)/n})$ if $\sigma \geq \sqrt{(\log n)/n}$, and $O((\log n)/(n\sigma))$ otherwise. Kozma et al. also show, in the same setting, that the expected sum of the squares of all Delaunay edge lengths is $O(1)$. In [9] the authors consider the Delaunay triangulation of an infinite random (Poisson) point set in d dimensional space. In particular, they study different properties of the subset of those Delaunay edges completely included in a cube $[0, n^{1/d}] \times \cdots \times [0, n^{1/d}]$. For the maximum length of a Delaunay edge in this setting, they observe that in expectation is in $\Theta(\log^{1/d} n)$.

The lengths of longest edges in geometric graphs induced by random point sets has also been studied for graphs related to the Delaunay graph, including Gabriel graphs [10] and relative neighborhood (RNG) graphs [11,12]. In particular, Wan and Yi [10] show that for n points uniformly distributed in a unit-area disk, the ratio of the length of a longest Gabriel edge to $\sqrt{(\ln n)/(\pi n)}$ is asymptotically almost surely equal to 2, and the expected number of "long" Gabriel edges, of length at least $2\sqrt{(\ln n+\xi)/(\pi n)}$, is asymptotically almost surely equal to $2e^{-\xi}$, for any fixed ξ. In [13], while studying the maximum degree of Gabriel and Yao graphs, the authors observe that the probability that the maximum edge length is greater than $3\sqrt{(\log n)/n}$ tends to zero, a bound that they claim becomes $O(((\log n)/n)^{1/d})$ for d dimensions. An overview of related problems can be found in [14].

Interest in bounding the length of a longest Delaunay edge in two-dimensional spaces has grown out of extensive algorithmic work [15–17] aimed at reducing the energy consumption of geographically routing messages in Radio Networks. Multidimensional Delaunay graphs are well studied in computational geometry from the point of view of efficient algorithms to construct them (see [3] and references therein), but only limited results are known regarding probabilistic analysis of Delaunay graphs in higher dimensions [18].

Overview of our results We study the probabilistic length of longest Delaunay edges for points distributed uniformly in geometric domains in two and more dimensions. Since the length of the longest Delaunay edge is strongly influenced by the boundary of the enclosing region, we study the problem for two cases, which we call *with boundary* and *without boundary*.

Our results include upper and lower bounds for d-dimensional bodies with and without boundaries, that hold for a parametric error probability ε and are computed up to the constant factors (they are tight only asymptotically). In comparison, the upper bounds presented in [6] are only asymptotic, are restricted to two dimensions ($d=2$), and apply to domains with boundary (disks), although results without boundary are implicitly given, since the results are parametric in the distance to the boundary.

All our bounds apply for any $d>1$. The asymptotic results, shown in Table 2, are tight for $e^{-cn} \leq \varepsilon \leq n^{-c}$, for any constant $c>0$, and $d \in O(1)$. As it can be seen in Table 1 where the results are denoted asymptotically for readability. To the best of our knowledge, this is the first comprehensive study of this problem.

The precise results obtained are detailed in Table 2. (Refer to Section 2 for necessary notation.) In order to compare upper and lower bounds for bodies with boundary, it is crucial to notice that we bound the volume of a circular segment (2D) and the volume of a ball cap (3D), which can be approximated by polynomials of third and fourth degree, respectively, on the diameter of the base. Upper bounds are proved exploiting the fact that, thanks to the uniform density, it is very unlikely that a "large" volume is void of points. Lower bounds, on the other hand, are proved by showing that a configuration that yields a Delaunay edge of a certain length is not very unlikely.

In the following section, some necessary notation is introduced. Upper and lower bounds for enclosing bodies without boundaries are shown in Section 3, and the case with boundaries is covered in Section 4.

2. Preliminaries

The following notation will be used throughout. We will restrict attention to Euclidean (L_2) spaces. A *d-sphere*, $S=S_{r,c}$, of radius r is the set of all points in a $(d+1)$-dimensional space that are located at distance r (the *radius*) from a given point c (the *center*). A *d-ball*, $B=B_{r,c}$, of radius r is the set of all points in a d-dimensional space that are located at distance *at most* r (the *radius*) from a given point c (the *center*). The *area* of a d-sphere S (in $(d+1)$-space) is its d-dimensional volume. The *volume* of a d-ball B (in d-space) is its d-dimensional volume. We refer to a *unit sphere* as a sphere of area 1 and a *unit ball* as a ball of volume 1. (This is in contrast with the definition of a "unit" ball/sphere as a unit-radius ball/sphere. In particular, notice that in our definition the unit sphere is *not* the boundary of a unit ball. We find it convenient to standardize the volume/area to be 1 in all dimensions.)

Table 2
Summary of results. α_2, α_3 are constants. $\alpha(d)$ and $\kappa_2(d)$ are functions of d.

	d	Upper bound: w.p. $\geq 1-\varepsilon$, $\nexists \widehat{ab} \in D(P)$	Lower bound: w.p. $\geq \varepsilon$, $\exists \widehat{ab} \in D(P)$
Without boundary	d	$A_d(\delta(a,b)) \geq \frac{\ln(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon)}{n-d-1}$	$A_d(\delta(a,b)) \geq \frac{\ln((e-1)/(e^2\varepsilon))}{n-2+\ln((e-1)/(e^2\varepsilon))}$
	1	$\delta(a,b) \geq \frac{\ln(\binom{n}{2}/\varepsilon)}{n-2}$	$\delta(a,b) \geq \frac{\ln((e-1)/(e^2\varepsilon))}{n-2+\ln((e-1)/(e^2\varepsilon))}$
	2	$\delta(a,b) \geq \frac{\cos^{-1}(1-\frac{2\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3})}{\sqrt{\pi}}$	$\delta(a,b) \geq \frac{\cos^{-1}(1-\frac{2\ln((e-1)/(e^2\varepsilon))}{n-2+\ln((e-1)/(e^2\varepsilon))})}{\sqrt{\pi}}$
With boundary	d	$V_d(d(a,b)) \geq \frac{\ln(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon)}{n-d-1}$	$d(a,b) \geq \rho_1/\sqrt{d-1}$: $V_d(\rho_1) = \frac{\ln(\alpha(d)/\varepsilon)}{\kappa_2(d)(n-2+\ln(\alpha(d)/\varepsilon))}$
	2	$d(a,b) \geq \sqrt[3]{\frac{16}{\sqrt{\pi}} \frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}}$	$d(a,b) \geq \sqrt[3]{\frac{4}{7\sqrt{\pi}} \frac{\ln(\alpha_2/\varepsilon)}{n-2+\ln(\alpha_2/\varepsilon)}}$
	3	$d(a,b) \geq \sqrt[4]{\frac{96}{\pi^{3/2}} \frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}}$	$d(a,b) \geq \sqrt[4]{\frac{64\sqrt[3]{6}}{83\pi^{4/3}} \frac{\ln(\alpha_3/\varepsilon)}{n-2+\ln(\alpha_3/\varepsilon)}}$

Let P be a set of points on a d-sphere, S. Given two points $a, b \in P$, let $\widehat{ab}$ be the arc of a great circle between them. Let $\delta(a,b)$ be the length of the arc $\widehat{ab}$, which is also known as the *orthodromic distance* between a and b on the sphere S. Let the *orthodromic diameter* of a subset $X \subseteq S$ be the greatest orthodromic distance between a pair of points in X. A *spherical cap on S* is the set of all points at orthodromic distance at most r from some center point $c \in S$. Let $A_d(x)$ be the area (d-volume) of a spherical cap of orthodromic diameter x, on a d-sphere of surface area 1. A *ball cap of B* is the intersection of a d-ball B with a closed halfspace, bounded by a hyperplane h, in d-space; the *base* of a ball cap is the $(d-1)$-ball that is the intersection of h with the ball B. Let $V_d(x)$ be the d-volume of a ball cap of base diameter x, of a d-ball of volume 1. For any pair of points a, b, let $d(a,b)$ be the Euclidean distance between a and b, i.e. $d(a,b) = \|\overrightarrow{ab}\|_2$. Let $D(P)$ be the Delaunay graph of a set of points P.

The following definitions of a Delaunay graph, $D(P)$, of a finite set P of points in a d-dimensional body follow the standard definitions of Delaunay graphs (see, e.g., Theorem 9.6 in [3]).

Definition 1. Let P be a generic set of points on a d-sphere S.

(i) A set $F \subseteq P$ of $d+1$ points define the vertices of a *Delaunay face* of $D(P)$ if and only if there is a d-dimensional spherical cap $C \subset S$ such that F is contained in the boundary, ∂C, of C and no points of P lie in the interior of C (relative to the sphere S).
(ii) Two points $a, b \in P$ form a *Delaunay edge*, an arc of $D(P)$, if and only if there is a d-dimensional spherical cap C such that $a, b \in \partial C$ and no points of P lie in the interior of C (relative to the sphere S).

Definition 2. Let P be a generic set of points in a d-ball B.

(i) A set $F \subseteq P$ of $d+1$ points define the vertices of a *Delaunay face* of $D(P)$ if and only if there is a d-ball B' such that F is contained in the boundary, $\partial B'$, of B' and no points of P lie in the interior of B'.
(ii) Two points $a, b \in P$ form a *Delaunay edge*, an arc of $D(P)$, if and only if there is a d-ball B' such that $a, b \in \partial B'$ and no points of P lie in the interior of B'.

The following inequalities [19] are used throughout

$$e^{-x/(1-x)} \leq 1-x \leq e^{-x}, \quad \text{for } 0 < x < 1. \tag{1}$$

3. Enclosing body without boundary

The following theorems show upper and lower bounds on the length of arcs in the Delaunay graph on a d-sphere.

3.1. Upper bound

Theorem 1. *Consider the Delaunay graph $D(P)$ of a set P of n points in dimension $d \geq 1$, where $n \geq d+2$, distributed uniformly and independently at random in a unit d-sphere, S. Then, for $0 < \varepsilon < 1$, the probability is at least $1-\varepsilon$ that there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$A_d(\delta(a,b)) \geq \frac{\ln(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon)}{n-d-1}. \tag{$*$}$$

Proof. Let E_ε be the event that "there exists an arc $\widehat{ab} \in D(P)$, $a, b \in P$, with inequality $(*)$ satisfied." Our goal is to prove that $P(E_\varepsilon) \le \varepsilon$.

Let us consider a fixed pair of points, $a, b \in P$. We let $E_{a,b}$ be the event that $\widehat{ab} \in D(P)$. For any subset $Q \subset P$ of $d+1$ points containing a and b, let C_Q denote the spherical cap through Q and let F_Q denote the event that the interior of C_Q contains no points of P (i.e., $int(C_Q) \cap P = \emptyset$).

Thus, we can write $E_{a,b} = \bigcup_Q F_Q$ as the union, over all $\binom{n-2}{(d+1)-2} = \binom{n-2}{d-1}$ subsets $Q \subset P$ with $|Q| = d+1$ and $a, b \in Q$, of the events F_Q. Then, by the union bound, we know that $P(E_{a,b}) \le \sum_Q P(F_Q)$. Further, in order for event F_Q to occur, all points of P *except* the $d+1$ points of Q must lie *outside* the spherical cap C_Q through Q; thus, $P(F_Q) = (1 - \mu_d(C_Q))^{n-(d+1)}$, where $\mu_d(C_Q)$ denotes the d-volume of C_Q.

We see that $P(F_Q) \le (1 - A_d(\delta(a,b)))^{n-(d+1)}$, since, for any subset $Q \supset \{a, b\}$, the d-volume $\mu_d(C_Q)$ is at least as large as the d-volume, $A_d(\delta(a,b))$, of the spherical cap having orthodromic diameter $\delta(a,b)$. In other words, $A_d(\delta(a,b))$ is the d-volume of the smallest volume spherical cap whose boundary passes through a and b. This property can be seen by noticing that, fixing a spherical cap, the largest arc is an orthodromic diameter. Hence, fixing the arc $\widehat{ab}$, the smallest spherical cap whose boundary passes through a and b has orthodromic diameter $\delta(a,b)$.

Altogether, we get

$$\begin{aligned} P(E_{a,b}) &\le \sum_Q P(F_Q) = \sum_Q \left(1 - \mu_d(C_Q)\right)^{n-(d+1)} \\ &\le \binom{n-2}{d-1}\left(1 - A_d\left(\delta(a,b)\right)\right)^{n-(d+1)}. \end{aligned}$$

Now, the event of interest is

$$E_\varepsilon = \bigcup_{a,b \in P:(*) \text{ holds}} E_{a,b}.$$

The inequality $(*)$ is equivalent to

$$(n-d-1)A_d\left(\delta(a,b)\right) \ge \ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right),$$

which is equivalent to

$$\left(e^{-A_d(\delta(a,b))}\right)^{(n-d-1)} \le \frac{\varepsilon}{\binom{n}{2}\binom{n-2}{d-1}}.$$

Since, by Inequality (1), $e^{-x} \ge 1 - x$, the above inequality implies that

$$\left(1 - A_d\left(\delta(a,b)\right)\right)^{(n-d-1)} \le \frac{\varepsilon}{\binom{n}{2}\binom{n-2}{d-1}},$$

which implies that

$$\binom{n}{2}\binom{n-2}{d-1}\left(1 - A_d\left(\delta(a,b)\right)\right)^{(n-d-1)} \le \varepsilon.$$

Using the union bound, we get

$$P(E_\varepsilon) = P\left(\bigcup_{a,b \in P:(*) \text{ holds}} E_{a,b}\right) \le \sum_{a,b \in P:(*) \text{ holds}} P(E_{a,b}).$$

Since each term $P(E_{a,b})$ in the above summation is bounded above by $\binom{n-2}{d-1}(1 - A_d(\delta(a,b)))^{n-(d+1)}$, and there are at most $\binom{n}{2}$ terms in the summation, we get

$$\begin{aligned} P(E_\varepsilon) &\le \sum_{a,b \in P:(*) \text{ holds}} P(E_{a,b}) \\ &\le \binom{n}{2}\binom{n-2}{d-1}\left(1 - A_d\left(\delta(a,b)\right)\right)^{(n-d-1)} \le \varepsilon. \qquad \square \end{aligned}$$

The following corollaries for $d = 1$ and $d = 2$ can be obtained from Theorem 1 using the corresponding surface areas.

Corollary 1. *In the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random on a unit circle (1-sphere), with probability at least $1-\varepsilon$, for $0<\varepsilon<1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{\ln(\binom{n}{2}/\varepsilon)}{n-2}.$$

Corollary 2. *In the Delaunay graph $D(P)$ of a set P of $n > 3$ points distributed uniformly and independently at random on a unit sphere (2-sphere), with probability at least $1-\varepsilon$, for $0<\varepsilon<1$, there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{1}{\sqrt{\pi}} \cos^{-1}\left(1 - \frac{2\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}\right).$$

Proof. The radius of a unit 2-sphere is $R = 1/(2\sqrt{\pi})$. Thus, the surface area of a spherical cap of a 2-sphere is $2\pi Rh = \sqrt{\pi}h$, where h is the height of the cap. On the other hand, the central angle of a cap with orthodromic diameter ρ is $2\pi\rho/\sqrt{\pi} = 2\sqrt{\pi}\rho$. Thus, the height is $h = 1/(2\sqrt{\pi})(1-\cos(\sqrt{\pi}\rho))$. This yields that the surface area of a spherical cap of a 2-sphere whose orthodromic diameter is ρ is $(1-\cos(\sqrt{\pi}\rho))/2$. Replacing in Theorem 1, the claim follows. □

3.2. Lower bound

Theorem 2. *Consider the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-sphere, S. Then, for any $0<\varepsilon<1$ and ρ such that*

$$A_d(\rho) = \frac{\ln((e-1)/(e^2\varepsilon))}{n-2+\ln((e-1)/(e^2\varepsilon))}, \quad \text{and}$$
$$A_d(2\rho) \leq 1 - 1/(n-1),$$

the probability is at least ε that there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that $A_d(\delta(a,b)) \geq A_d(\rho)$.

Proof. To see that the claim is not vacuously true, fix d and let $A_d(2\rho) = f(d)A_d(\rho)$, for some function $f(\cdot)$. Then, we want to show that $A_d(2\rho) = f(d)\ln((e-1)/(e^2\varepsilon))/(n-2+\ln((e-1)/(e^2\varepsilon))) \leq 1-1/(n-1)$ for some $0<\varepsilon<1$. This is true for $\varepsilon \geq (e-1)/(e^2\exp((n-2)^2/(1+(n-1)(f(d)-1))))$.

In order to prove the claim, we consider a configuration given by a specific pair of points and a specific empty spherical cap circumscribing them, that would yield a Delaunay arc between those points. Then, we relate the probability of existence of such a configuration to the distance between the points. Finally, we relate this quantity to the desired parametric probability. The details follow.

For any pair of points $a, b \in P$, by Definition 1, for the arc $\widehat{ab}$ to be in $D(P)$, there must exist a d-dimensional spherical cap C such that a and b are located on the boundary of the cap base, and the cap surface of C is void of points from P. We compute the probability of such an event as follows.

Let $\rho' > \rho$ be such that $A_d(2\rho') - A_d(2\rho) = 1/(n-1)$. Such a value ρ' exists because $A_d(2\rho) \leq 1-1/(n-1)$. Consider any point $a \in P$. The probability, p_1, that some other point b is located so that $\rho < \delta(a,b) \leq \rho'$ can be computed by considering the spherical annulus centered at a with ρ (resp., ρ') equal to the minimum (resp., maximum) orthodromic distance to a (i.e., we consider the difference between a spherical cap of orthodromic diameter $2\rho'$ and a spherical cap of orthodromic diameter 2ρ). Then, $p_1 = 1-(1-1/(n-1))^{n-1} \geq 1-1/e$, by Inequality (1).

The spherical cap with orthodromic diameter $\delta(a,b)$ is empty with probability $(1-A_d(\delta(a,b)))^{n-2}$. To lower bound this probability we consider separately the spherical cap with orthodromic diameter ρ and the remaining annulus of the spherical cap with orthodromic diameter $\delta(a,b)$. The probability that the annulus is empty, call it p_2, is lower bounded by upper bounding the area $A_d(\delta(a,b)) - A_d(\rho) \leq A_d(\rho') - A_d(\rho) \leq A_d(2\rho') - A_d(2\rho) = 1/(n-1)$. Then, $p_2 \geq (1-1/(n-1))^{n-2} \geq 1/e$, by Inequality (1).

Finally, the probability that the spherical cap with orthodromic diameter ρ is empty, call it p_3, is, by Inequality (1),

$$p_3 = \left(1-A_d(\rho)\right)^{n-2} \geq \exp\left(-\frac{A_d(\rho)(n-2)}{1-A_d(\rho)}\right)$$
$$= \exp\left(-\ln\left(\frac{e-1}{e^2\varepsilon}\right)\right) = \frac{e^2\varepsilon}{e-1}.$$

Therefore,

$$Pr\left(\widehat{ab} \in D(P)\right) \geq p_1p_2p_3 \geq \left(1-\frac{1}{e}\right)\frac{1}{e}\frac{e^2\varepsilon}{e-1} = \varepsilon. \qquad \square$$

The following corollaries for $d=1$ and $d=2$ can be obtained from Theorem 2 using the corresponding surface areas.

Corollary 3. *In the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random on a unit circle (1-sphere), with probability at least ε, for any $(e-1)/\exp(n+4/n) \le \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \ge \frac{\ln((e-1)/(e^2\varepsilon))}{n-2+\ln((e-1)/(e^2\varepsilon))}.$$

Proof. The lower bound on $\delta(a,b)$ can be obtained by replacing in Theorem 2 the surface of the spherical cap, which for $d=1$ is the length of the arc. Regarding the lower bound on ε, in the proof of Theorem 2, it was shown that the conditions of the theorem can be met by imposing a lower bound on ε that depends on d. Using $d=1$, we obtain that $f(d)=2$ and $\varepsilon \ge (e-1)/(e^2\exp((n-2)^2/(1+(n-1)(f(d)-1)))) = (e-1)/\exp(n+4/n)$. □

Corollary 4. *In the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit sphere (2-sphere), with probability at least ε, for any $e^{-n+2\sqrt{n-1}-1} \le \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \ge \frac{1}{\sqrt{\pi}}\cos^{-1}\left(1-\frac{2\ln((e-1)/(e^2\varepsilon))}{n-2+\ln((e-1)/(e^2\varepsilon))}\right).$$

Proof. As shown in the proof of Corollary 2, the surface area of a spherical cap of a 2-sphere whose orthodromic diameter is ρ is $(1-\cos(\sqrt{\pi}\rho))/2$. Replacing in Theorem 2, the claim follows. □

4. Enclosing body with boundary

The following theorems show upper and lower bounds on the lengths of edges in the Delaunay graph in a d-ball. (Recall that we refer to a unit ball as a ball of volume 1.)

4.1. Upper bound

Theorem 3. *Consider the Delaunay graph $D(P)$ of a set P of $n > d+1 \ge 2$ points distributed uniformly and independently at random in a unit d-ball, B. Then, for $0 < \varepsilon < 1$, the probability is at least $1-\varepsilon$ that there is no edge $(a,b) \in D(P)$, $a, b \in P$, such that*

$$V_d\big(d(a,b)\big) \ge \frac{\ln(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon)}{n-d-1}. \qquad (**)$$

Proof. In order to prove this claim, we consider any one set of $d+1$ points in P. Then, we relate the probability that the ball circumscribing the set is empty, to the distance separating the points. Finally, we combine the probabilities for all possible pairs of points and sets and we relate this quantity to the desired parametric probability. The details follow.

Let E_ε be the event that "there exists an edge $(ab) \in D(P)$, $a, b \in P$, with inequality (**) satisfied" Our goal is to prove that $P(E_\varepsilon) \le \varepsilon$.

Let us consider a fixed pair of points, $a, b \in P$. We let $E_{a,b}$ be the event that $(ab) \in D(P)$. For any subset $Q \subset P$ of $d+1$ points containing a and b, let B_Q denote the ball through Q and let F_Q denote the event that the interior of B_Q contains no points of P (i.e., $int(B_Q) \cap P = \emptyset$).

Thus, we can write $E_{a,b} = \bigcup_Q F_Q$ as the union, over all $\binom{n-2}{(d+1)-2} = \binom{n-2}{d-1}$ subsets $Q \subset P$ with $|Q| = d+1$ and $a, b \in Q$, of the events F_Q. Then, by the union bound, we know that $P(E_{a,b}) \le \sum_Q P(F_Q)$. Further, in order for event F_Q to occur, all points of P *except* the $d+1$ points of Q must lie *outside* the ball B_Q through Q; thus, $P(F_Q) = (1-\mu_d(B_Q \cap B))^{n-(d+1)}$, where $\mu_d(B_Q \cap B)$ denotes the d-volume of $B_Q \cap B$. (Recall that points lie only inside B.)

We see that $P(F_Q) \le (1-V_d(d(a,b)))^{n-(d+1)}$, since, for any subset $Q \supset \{a,b\}$, the d-volume $\mu_d(B_Q \cap B)$ is at least as large as the d-volume, $V_d(d(a,b))$, of the ball cap of B having base diameter $d(a,b)$. In other words, $V_d(d(a,b))$ is the d-volume of the smallest volume ball cap of B whose base boundary passes through a and b. This property can be seen by noticing that, fixing a ball cap, the largest segment in the base is a diameter. Hence, fixing a segment (a,b), the smallest ball cap whose boundary passes through a and b has base diameter $d(a,b)$.

Altogether, we get

$$P(E_{a,b}) \le \sum_Q P(F_Q) = \sum_Q \big(1-\mu_d(B_Q \cap B)\big)^{n-(d+1)}$$
$$\le \binom{n-2}{d-1}\big(1-V_d\big(d(a,b)\big)\big)^{n-(d+1)}.$$

Now, the event of interest is

$$E_\varepsilon = \bigcup_{a,b \in P:(**)\text{ holds}} E_{a,b}.$$

The inequality (∗∗) is equivalent to

$$(n-d-1)V_d\big(d(a,b)\big) \geq \ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right),$$

which is equivalent to

$$\left(e^{-V_d(d(a,b))}\right)^{(n-d-1)} \leq \frac{\varepsilon}{\binom{n}{2}\binom{n-2}{d-1}}.$$

Since, by Inequality (1), $e^{-x} \geq 1-x$, the above inequality implies that

$$\big(1-V_d\big(d(a,b)\big)\big)^{(n-d-1)} \leq \frac{\varepsilon}{\binom{n}{2}\binom{n-2}{d-1}},$$

which implies that

$$\binom{n}{2}\binom{n-2}{d-1}\big(1-V_d\big(d(a,b)\big)\big)^{(n-d-1)} \leq \varepsilon.$$

Using the union bound, we get

$$P(E_\varepsilon) = P\left(\bigcup_{a,b\in P:(**)\text{ holds}} E_{a,b}\right) \leq \sum_{a,b\in P:(**)\text{ holds}} P(E_{a,b}).$$

Since each term $P(E_{a,b})$ in the above summation is bounded above by $\binom{n-2}{d-1}(1-V_d(d(a,b)))^{n-(d+1)}$, and there are at most $\binom{n}{2}$ terms in the summation, we get

$$\begin{aligned}P(E_\varepsilon) &\leq \sum_{a,b\in P:(**)\text{ holds}} P(E_{a,b})\\ &\leq \binom{n}{2}\binom{n-2}{d-1}\big(1-V_d\big(d(a,b)\big)\big)^{(n-d-1)} \leq \varepsilon. \qquad \square\end{aligned}$$

The following corollaries for $d=2$ and $d=3$ can be obtained from Theorem 3 using the corresponding volumes.

Corollary 5. *In the Delaunay graph $D(P)$ of a set P of $n>3$ points distributed uniformly and independently at random in a unit disk (2-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}(n-2)e^{-\sqrt{2}(n-3)/\pi} < \varepsilon < 1$, there is no edge $(a,b)\in D(P)$, $a,b\in P$, such that*

$$d(a,b) \geq \sqrt[3]{\frac{16}{\sqrt{\pi}}\frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}}.$$

Proof. Consider the intersection of the radius of the unit disk perpendicular to (a,b) with the circumference of the unit disk, call this point x. The area of the triangle $\triangle abx$ is a strict lower bound on $V_2(d(a,b))$. From Theorem 3, we have the condition

$$V_2\big(d(a,b)\big) \geq \frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}.$$

Thus, it is enough to show that

$$\frac{d(a,b)}{2}\left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{1}{\pi} - \frac{d(a,b)^2}{4}}\right) \geq \frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}.$$

Making $\rho = d(a,b)\sqrt{\pi}/2$, we want

$$\sqrt{\rho^2-\rho^4} \leq \rho - \pi\frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}. \tag{2}$$

If $d(a,b) < 2\sqrt{\pi}\ln(\binom{n}{2}(n-2)/\varepsilon)/(n-3)$, there is nothing to prove because

$$\frac{2\sqrt{\pi}\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3} < \sqrt[3]{\frac{16\ln(\binom{n}{2}(n-2)/\varepsilon)}{\sqrt{\pi}(n-3)}},$$

for any $\varepsilon > \binom{n}{2}(n-2)\exp(-\sqrt{2}(n-3)/\pi)$. Otherwise, we have that $\rho \geq \pi \ln(\binom{n}{2}(n-2)/\varepsilon)/(n-3)$, and by squaring both sides of (2) we get

$$\rho^4 \geq 2\rho\pi \frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3} - \left(\pi \frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}\right)^2,$$

which is implied by

$$\rho^3 \geq 2\pi \frac{\ln(\binom{n}{2}(n-2)/\varepsilon)}{n-3}.$$

Substituting $\rho = d(a,b)\sqrt{\pi}/2$ into the above inequality, the claim follows. □

Corollary 6. *In the Delaunay graph $D(P)$ of a set P of $n > 4$ points distributed uniformly and independently at random in a unit ball (3-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}\binom{n-2}{2}e^{-2(n-4)/(3\sqrt{\pi})} < \varepsilon < 1$, there is no edge $(a,b) \in D(P)$, $a, b \in P$, such that*

$$d(a,b) \geq \sqrt[4]{\frac{96}{\pi^{3/2}} \frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}}.$$

Proof. Consider the intersection of the radius of the unit ball perpendicular to (a,b) with the surface of the unit ball, call this point d. The volume of the cone whose base is the disk whose diameter is (a,b) and its vertex is d is a strict lower bound on $V_2(d(a,b))$. From Theorem 3, we have the condition

$$V_3\big(d(a,b)\big) \geq \frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}.$$

Thus, it is enough to show that

$$\frac{\pi}{3}\left(\frac{d(a,b)}{2}\right)^2\left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{1}{\pi} - \frac{d(a,b)^2}{4}}\right) \geq \frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}.$$

Making $\rho = d(a,b)\sqrt{\pi}/2$, we want

$$\sqrt{\rho^4 - \rho^6} \leq \rho^2 - 3\sqrt{\pi}\frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}. \tag{3}$$

If $d(a,b) < \sqrt{12\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)/(\sqrt{\pi}(n-4))}$, there is nothing to prove because

$$\sqrt{\frac{12\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{\sqrt{\pi}(n-4)}} < \sqrt[4]{\frac{96}{\pi^{3/2}} \frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}},$$

for any $\varepsilon > \binom{n}{2}\binom{n-2}{2}\exp(-2(n-4)/(3\sqrt{\pi}))$. Otherwise, we have that $\rho^2 \geq 3\sqrt{\pi}\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)/(n-4)$, and by squaring both sides of (3) we get

$$\rho^6 \geq 6\rho^2\sqrt{\pi}\frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4} - \left(3\sqrt{\pi}\frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}\right)^2,$$

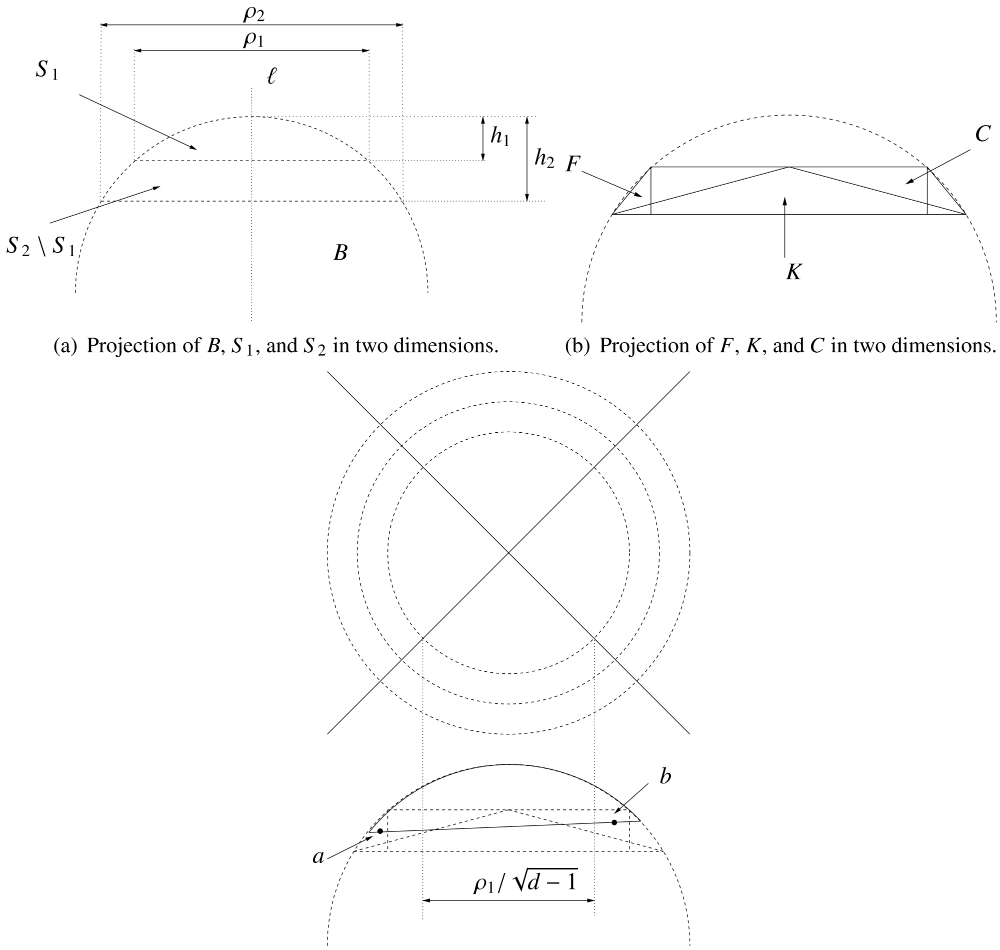
which is implied by

$$\rho^4 \geq 6\sqrt{\pi}\frac{\ln(\binom{n}{2}\binom{n-2}{2}/\varepsilon)}{n-4}.$$

Substituting $\rho = d(a,b)\sqrt{\pi}/2$ into the above inequality, the claim follows. □

4.2. Lower bound

As in the case without boundary, we prove our lower bound by showing a configuration given by a specific pair of points and a specific empty body circumscribing them, that would yield a Delaunay edge between those points. Then, we relate the probability of existence of such configuration to the distance between the points and to the desired parametric probability.



(a) Projection of B, S_1, and S_2 in two dimensions.

(b) Projection of F, K, and C in two dimensions.

(c) Points in $F \setminus K \cup C$ for $d = 3$ projected in two dimensions.

Fig. 1. Illustration of Theorem 4.

Theorem 4. *For any $d > 1$, let*

$$\alpha(d) = \left(1 - e^{-\kappa_1(d)/\kappa_2(d)}\right)\left(1 - e^{-\kappa_1(d)/(2\kappa_2(d)(2d-2))}\right)$$

$$\kappa_1(d) = \frac{1}{d-1}\sum_{i=0}^{d-2}\left(\left(\frac{d}{\sqrt{d^2-1}}\right)^i - \frac{\sqrt{d^2-1}}{d}\right)$$

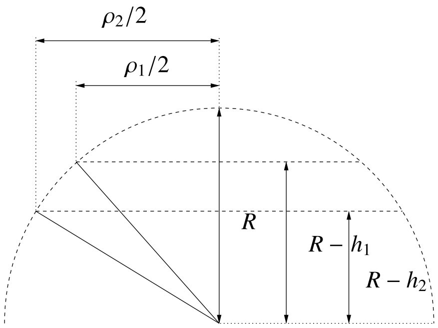
$$\kappa_2(d) = \left(1 + \left(\frac{2d-1}{d-1}\right)^{d-1}\frac{d}{d-1}\right).$$

For any $n > 1$ and $0 < \varepsilon \le \alpha(d)/e$, given the Delaunay graph $D(P)$ of a set P of n points distributed uniformly and independently at random in a unit d-ball, with probability at least ε, there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that $d(a, b) \ge \rho_1/\sqrt{d-1}$, where

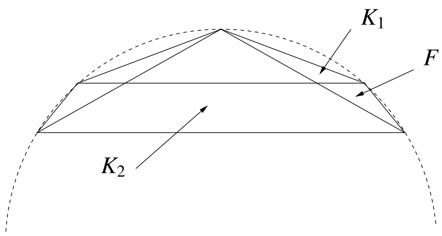
$$V_d(\rho_1) = \frac{\ln(\alpha(d)/\varepsilon)}{\kappa_2(d)(n - 2 + \ln(\alpha(d)/\varepsilon))}.$$

Proof. We illustrate the proof in Figs. 1 and 2. Throughout the proof, we refer to a body and its set of space points with the same name indistinctly. Let $V(X)$ be the volume of a body (or a set of space points) X. Let the unit ball where points are sampled be called B. Consider two ball caps of B, concentric on a line ℓ, called S_1 and S_2, with bases B_1 and B_2 of diameters ρ_1 and ρ_2, and heights h_1 and h_2 respectively (see Fig. 1(a)). Inside $S_2 \setminus S_1$, consider the following d-dimensional bodies of height $h_2 - h_1$: a cylinder C with base B_1; a cone K of base B_2; and a frustum F of bases B_2 and B_1 (see Fig. 1(b)).

Consider the body $F \setminus (C \cup K)$ evenly partitioned into $2(d-1)$ pieces such that two of them, call them B_a and B_b, have the following property. For any pair of points $a \in B_a$ and $b \in B_b$, the points a and b are separated by a distance of at least



(a) Ratio of diameters.



(b) Projection of F, K_1, and K_2 in two dimensions.

Fig. 2. Illustration of Theorem 4.

$\rho_1/\sqrt{d-1}$. To see why such a partition exists, consider a $(d-1)$-dimensional cube, call it C_1, inscribed in the base of S_1. The maximum diagonal of C_1 has length ρ_1, and, hence, each side of C_1 has length $\rho_1/\sqrt{d-1}$.

Additionally, we observe that, for any pair of points $a \in B_a$ and $b \in B_b$, there exists a ball cap S that contains the points a and b in its base of diameter ρ such that $V_d(\rho) \le V_d(\rho_2)$. To see why the latter is true, consider the following. Without loss of generality assume that the point a is closer to B_2 than b. Then, consider a 2-dimensional plane h containing the line ℓ and the point a and the projection of b on h. On h, the point closest to B_2 is located above the projection of K (see Fig. 1(c)).

If S is void of points, the configuration described implies the existence of an empty d-ball of infinite radius with a and b in its surface which proves that $(a,b) \in D(P)$. In the following, we show that such configuration occurs with big enough probability.

Let ρ_1 be such that $V_d(\rho_1)$ is as defined in the statement of the theorem. Let h_2 be such that $V(C) = dV_d(\rho_1)/(d-1)$. Let $q = \rho_2/\rho_1$. First, we prove upper and lower bounds on q to be used later.

Claim 7. $d/\sqrt{d^2-1} \le q \le (2d-1)/(d-1)$.

Proof. From the volume of C, we know that $h_2/h_1 = 1 + V(C)/(h_1 V(B_1))$. Consider a cone with the same volume and base as S_1. The height of such cone, which is bigger than h_1, is $dV_d(\rho_1)/V(B_1)$. That is, $h_1 < dV_d(\rho_1)/V(B_1)$. Consider also a cylinder with the same volume and base as S_1. The height of such cylinder, which is smaller than h_1, is $V_d(\rho_1)/V(B_1)$. That is, $h_1 > V_d(\rho_1)/V(B_1)$. Replacing those bounds and using that the fact that $V(C) = dV_d(\rho_1)/(d-1)$, we get

$$\frac{d}{d-1} \le \frac{h_2}{h_1} \le \frac{2d-1}{d-1}. \tag{4}$$

Consider a 2-dimensional projection of the configuration described (see Fig. 2(a)). Let R be the radius of B. Then, using Pythagoras' theorem, $R^2 = (\rho_2/2)^2 + (R-h_2)^2 = (\rho_1/2)^2 + (R-h_1)^2$. Subtracting,

$$\begin{aligned} q^2 &= 1 + \frac{(R-h_1)^2 - (R-h_2)^2}{(\rho_1/2)^2} \\ &\ge 1 + \left(\frac{h_2}{h_1} - 1\right)\left(1 - \frac{1}{2 - h_1/h_2}\right) \\ &= \frac{h_2}{h_1(2 - h_1/h_2)}. \end{aligned}$$

Using Inequality (4),

$$q^2 \ge \frac{d}{d-1} \cdot \frac{1}{2-(d-1)/d} = \frac{d^2}{d^2-1}.$$

Which proves the lower bound. For the upper bound, consider the cones K_1 and K_2 inscribed in S_1 and S_2 respectively (see Fig. 2(b)). It can be seen that

$$V(K_1 \cup F) > V(K_2). \tag{5}$$

The volumes of K_1 and K_2 are

$$V(K_1) = \frac{h_1 V(B_1)}{d} = \frac{h_1 C(d-1)\rho_1^{d-1}}{d2^{d-1}}$$
$$V(K_2) = \frac{h_2 V(B_2)}{d} = \frac{h_1 C(d-1)\rho_1^{d-1}}{d2^{d-1}}.$$

Replacing in (5), the following inequality holds,

$$\rho_2^{d-1}\left(\rho_2 - \frac{h_2}{h_1}\rho_1\right) < \rho_1^{d-1}\left(\rho_2 - \frac{h_2}{h_1}\rho_1\right).$$

Given that $\rho_2^{d-1} > \rho_1^{d-1}$, it must be $\rho_2 < \rho_1 h_2/h_1$. Using Inequality (4), we have $q < (2d-1)/(d-1)$. □

For any $d > 1$, let $C(d) = \pi^{d/2}/\Gamma(1+d/2)$, where $\Gamma(\cdot)$ is the Gamma function. We compute the volume of $F \setminus (C \cup K)$ as $V(F) - V(C \cup K)$.

$$V(F) = C(d-1)\int_0^{h_2-h_1}\left(\rho_1/2 + \frac{\rho_2/2 - \rho_1/2}{h_2-h_1}z\right)^{d-1} dz$$
$$= \frac{V(C)}{d}\cdot\frac{q^d-1}{q-1}.$$

$$V(C \cup K) = C(d-1)\left((\rho_1/2)^{d-1}\int_0^{\rho_1(h_2-h_1)/\rho_2} dz + \int_{\rho_1(h_2-h_1)/\rho_2}^{(h_2-h_1)} r_K(z)^{d-1} dz\right)$$
$$= C(d-1)\left((\rho_1/2)^{d-1}\int_0^{\rho_1(h_2-h_1)/\rho_2} dz + \left(\frac{\rho_2/2}{h_2-h_1}\right)^{d-1}\int_{\rho_1(h_2-h_1)/\rho_2}^{(h_2-h_1)} z^{d-1} dz\right)$$
$$= V(C)\frac{1}{q}\left(1 + \frac{1}{d}(q^d-1)\right).$$

Thus,

$$V\big(F \setminus (C \cup K)\big) = V(C)\left(\frac{1}{d}\cdot\frac{q^d-1}{q-1} - \frac{1}{q}\left(1+\frac{1}{d}(q^d-1)\right)\right)$$
$$= \frac{V(C)}{d}\left(\frac{q^d-1}{q-1} - \frac{d+q^d-1}{q}\right)$$
$$= \frac{V(C)}{d}\left(\frac{q^d-1}{q-1} - q^{d-1} - \frac{d-1}{q}\right)$$
$$= \frac{V(C)}{d}\left(\frac{q^{d-1}-1}{q-1} - \frac{d-1}{q}\right)$$
$$= \frac{V(C)}{d}\sum_{i=0}^{d-2}\left(q^i - \frac{1}{q}\right). \tag{6}$$

Using Claim 7 and the fact that $V(C) = dV_d(\rho_1)/(d-1)$ in Eq. (6), $V(F \setminus (C \cup K)) \geq \kappa_1(d)V_d(\rho_1)$. Given that $\varepsilon \leq \alpha(d)/e$, we know that $V_d(\rho_1) \geq 1/(\kappa_2(d)n)$, then $V(F \setminus (C \cup K)) \geq \kappa_1(d)/(\kappa_2(d)n)$. Then, the probability that $F \setminus (C \cup K)$ contains at least one point of P is at least $1 - (1 - \kappa_1(d)/(\kappa_2(d)n))^n \geq 1 - e^{-\kappa_1(d)/\kappa_2(d)}$. Consider the body $F \setminus (C \cup K)$ evenly partitioned into $2(d-1)$ parts. The probability that any given one of these parts of $F \setminus (C \cup K)$ contains at least one point of $P \setminus \{a\}$, for some $a \in P$, is at least $1 - (1 - \kappa_1(d)/(\kappa_2(d)n(2d-2)))^{n-1} \geq 1 - e^{-\kappa_1(d)/(2\kappa_2(d)(2d-2))}$. Conditioned on the existence of two points $a, b \in P$ located as described earlier, let S be a ball cap of base B (of diameter ρ) such that B contains a and b and $S \subset S_2$ (see Fig. 1(c)). Such cap exists as shown before. The probability that S is void of points of P is lower bounded by upper bounding its volume. We know that $V(S) \leq V(S_2)$, and $V(S_2)$ can be upper bounded considering S_1 and $S_2 \setminus S_1$ separately, which we do as follows.

$$\begin{aligned}V(S_2) - V(S_1) &\le C(d-1)(\rho_2/2)^{d-1}(h_2 - h_1)\\ &\le C(d-1)\left(\frac{2d-1}{2(d-1)}\rho_1\right)^{d-1}(h_2 - h_1)\\ &= \left(\frac{2d-1}{d-1}\right)^{d-1}\frac{d}{d-1}V_d(\rho_1).\end{aligned}$$

Then $V(S) \le \kappa_2(d)V_d(\rho_1)$. Thus, the probability that S is empty is at least

$$\left(1 - \kappa_2(d)V_d(\rho_1)\right)^{n-2} \ge \exp\left(-\frac{\kappa_2(d)V_d(\rho_1)(n-2)}{1-\kappa_2(d)V_d(\rho_1)}\right).$$

Replacing, we get

$$\begin{aligned}Pr\big((a,b) \in D(P)\big) &\ge \alpha(d)\exp\left(-\frac{\kappa_2(d)V_d(\rho_1)(n-2)}{1-\kappa_2(d)V_d(\rho_1)}\right)\\ &= \varepsilon. \qquad \square\end{aligned}$$

Corollary 1. *For any $n > 1$ and $0 < \varepsilon \le \alpha/e$, where $\alpha = (1 - e^{-(2-\sqrt{3})/14})(1 - e^{-(2-\sqrt{3})/56})$, given the Delaunay graph $D(P)$ of a set P of n points distributed uniformly and independently at random in a unit circle, with probability at least ε, there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that*

$$d(a,b) \ge 2\sqrt[3]{\frac{\ln(\alpha/\varepsilon)}{14\sqrt{\pi}(n-2+\ln(\alpha/\varepsilon))}}.$$

Proof. Instantiating Theorem 4 in dimension $d = 2$, we know that with probability at least ε there is an edge $(a, b) \in D(P)$, such that $d(a, b) \ge \rho_1$, where

$$V_2(\rho_1) = \frac{\ln(\alpha/\varepsilon)}{7(n-2+\ln(\alpha/\varepsilon))}.$$

We upper bound the area of the circular segment of chord ρ_1 with the area of the rectangle circumscribing it.

$$V_2(\rho_1) \le \rho_1\left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{1}{\pi} - \frac{\rho_1^2}{4}}\right).$$

Hence,

$$\sqrt{\frac{\rho_1^2}{\pi} - \frac{\rho_1^4}{4}} \le \frac{\rho_1}{\sqrt{\pi}} - V_2(\rho_1).$$

Given that $\rho_1/\sqrt{\pi} \ge V_2(\rho_1)$, we can square both sides getting

$$\begin{aligned}\rho_1^4 &\ge 4\left(2\frac{\rho_1}{\sqrt{\pi}} - V_2(\rho_1)\right)V_2(\rho_1)\\ &\ge 4\frac{\rho_1}{\sqrt{\pi}}V_2(\rho_1), \quad \text{because } V_2(\rho_1) \le \rho_1/\sqrt{\pi}.\end{aligned}$$

Then we get $\rho_1/2 \ge \sqrt[3]{V_2(\rho_1)/(2\sqrt{\pi})}$ and replacing $V_2(\rho_1)$ the claim follows. $\square$

Corollary 2. *For any $n > 1$ and $0 < \varepsilon \le \alpha/e$, where $\alpha = (1 - e^{-\kappa_1(3)/\kappa_2(3)})(1 - e^{-\kappa_1(3)/(8\kappa_2(3))})$, $\kappa_1(3) = 1/2 - 7/(6\sqrt{8})$, and $\kappa_2(3) = 10 + 3/8$, given the Delaunay graph $D(P)$ of a set P of n points distributed uniformly and independently at random in a unit ball in $\mathbb{R}^3$, with probability at least ε, there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that*

$$d(a,b) \ge \sqrt{2}\sqrt[4]{\frac{\sqrt[3]{48/\pi^4}\ln(\alpha/\varepsilon)}{\kappa_2(3)(n-2+\ln(\alpha/\varepsilon))}}.$$

Proof. Instantiating Theorem 4 in $d = 3$, we know that with probability at least ε there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that $d(a, b) \ge \rho_1/\sqrt{2}$, where

$$V_3(\rho_1) = \frac{\ln(\alpha/\varepsilon)}{\kappa_2(3)(n-2+\ln(\alpha/\varepsilon))}.$$

We upper bound the volume of the ball cap of base diameter ρ_1 with the volume of the cylinder circumscribing it.

$$V_3(\rho_1) \le \frac{\pi\rho_1^2}{4}\left(\sqrt[3]{\frac{3}{4\pi}} - \sqrt{\left(\frac{3}{4\pi}\right)^{2/3} - \frac{\rho_1^2}{4}}\right).$$

Hence,

$$\sqrt{\left(\frac{\pi}{4}\sqrt[3]{\frac{3}{4\pi}}\right)^2 \rho_1^4 - \frac{\pi^2}{64}\rho_1^6} \le \frac{\pi\rho_1^2}{4}\sqrt[3]{\frac{3}{4\pi}} - V_3(\rho_1).$$

Given that $\pi\rho_1^2/4\sqrt[3]{3/(4\pi)} \ge V_3(\rho_1)$, we can square both sides getting

$$\begin{aligned}\frac{\pi^2}{64}\rho_1^6 &\ge \left(2\frac{\pi\rho_1^2}{4}\sqrt[3]{\frac{3}{4\pi}} - V_3(\rho_1)\right)V_3(\rho_1)\\ &\ge \frac{\pi\rho_1^2}{4}\sqrt[3]{\frac{3}{4\pi}}V_3(\rho_1), \quad \text{because } V_3(\rho_1) \le \pi\rho_1^2/4\sqrt[3]{3/(4\pi)}.\end{aligned}$$

Then we get $\rho_1/2 \ge \sqrt[4]{\sqrt[3]{48/\pi^4}V_3(\rho_1)}$ and replacing $V_3(\rho_1)$ the claim follows. □

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